

The Beltrami Model of De Sitter Space: From Snyder's quantized space-time to de Sitter invariant relativity*

Han-Ying Guo

Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, China.

In terms of the Beltrami model of de Sitter space we show that there is an interchangeable relation between Snyder's quantized space-time model in dS -space of momenta at the Planck length $\ell_P = (G\hbar c^{-3})^{1/2}$ and the dS -invariant special relativity in dS -spacetime of radius $R \simeq (3\Lambda^{-1})^{1/2}$, which is another fundamental length related to the cosmological constant. Here, the cosmological constant Λ is regarded as a fundamental constant together with the speed of light c , Newton constant G and Planck constant \hbar . Furthermore, the physics at two fundamental scales of length, the dS -radius R and the Planck length ℓ_P , should be dual to each other and linked via the gravity with local dS -invariance characterized by a dimensionless coupling constant $g = \sqrt{3}\ell_P/R \simeq (G\hbar c^{-3}\Lambda)^{1/2} \sim 10^{-61}$.

PACS numbers: 04.90.+e, 04.50+h, 03.30.+p, 02.40.Dr, 02.40.Gh

Contents

I. Introduction	2
II. Snyder's Quantized Space-time and DSR	3
III. The Beltrami Model	4
A. The Beltrami model of a Riemann sphere	4
B. The Beltrami model of dS -space	5
1. dS -hyperboloid $H_l \subset M^{1,4}$ and uniform 'great circular' motions	5
2. The Beltrami model of dS -space and uniform motions	6
C. Snyder's quantized space-time via Beltrami model	7
D. Klein's Erlangen program and the principle of relativity	7
IV. De Sitter Invariant Relativity	8
A. Inertial motions, transformations and the principle of relativity	8
B. Two kinds of simultaneity, the principle of relativity and cosmological principle	10
1. The Beltrami-time simultaneity	10
2. The Proper-time simultaneity and the Robertson-Walker-like coordinates	10
C. On temperature and entropy	11
V. The Planck Scale - Λ Duality	12
A. An interchangeable relation and the duality	12
B. Is the dimensionless constant calculable?	12
C. A simple model of dS -gravity	13
VI. Concluding Remarks	13
Acknowledgments	14

* Invited talk given at 'International workshop on noncommutative geometry and physics', Beijing, Nov. 7-10, 2005. To appear in the proceedings.

I. INTRODUCTION

Long time ago, Snyder proposed a quantized space-time model[1], in which the space-time coordinates are no longer ordinary real numbers rather some non-commutative operators. Snyder started with a projective geometry approach to the de Sitter (dS)-space of momenta with a large energy-momentum scale a near or at the Planck scale. Denoting the energy and momentum by means of the inhomogeneous projective coordinates, the space-time coordinates' operators \hat{x}_j are defined by 4-translation Killing vectors of $so(1, 4)$ -algebra. Thus, \hat{x}_j are noncommutative.

Recently, in order to explain the Greisen-Zatsepin-Kuz'min effects [2] the 'doubly spacial relativity' (DSR) models have been proposed [3]. In DSR, there is also a large energy-momentum scale κ near the Planck scale in addition to the speed of light c . It is found that there is a close relation between Snyder's model and DSR. In fact, DSR can be regarded as generalization of Snyder's model [4] and most DSR models with κ -Poincaré algebra can be realized geometrically by means of particular coordinate systems on dS , Minkowski ($Mink$) and anti- dS (AdS)-space of momenta [4] other than the inhomogeneous projective coordinates used by Snyder for the dS -space of momenta, or its counterpart in the AdS -space of momenta. Thus, in this point of view, there is a kind of coordinate transformations from Snyder's model to a kind of DSR in dS/AdS -space of momenta and vice versa, respectively.

For the dS -space the projective geometry model can be completely substituted by the Beltrami-like model (the Beltrami model for short) [5, 6] (see also [7]). It is very important that in either Beltrami coordinates or inhomogeneous projective ones the timelike and null geodesics in dS -space are in linear forms and all these properties are invariant under the fractional linear transformations with common denominator (\mathcal{FLT} s) of the dS -group $SO(1, 4)$. Thus, the particles and light signals move along these timelike and null geodesics, respectively, are all with *constant* coordinate velocities. Therefore, these particles and signals look like in free motion of inertia in a space without gravity. Namely, these motions should be regarded as a new kind of inertial motions in dS -space. For the definiteness, we focus on the Beltrami-coordinates and name the dS -space with such coordinates as the Beltrami- dS (BdS)-space.

Historically, de Sitter [9] first used the Beltrami coordinates for the spacetime of constant curvature in 1917, the same year he found his solution, in the course of the debate with Einstein on 'relative inertia'. A few years later, Pauli, in his famous book [10], noticed the Beltrami metric with Euclid signature, the Beltrami model of Riemann sphere, and asked for its physical application. However, there had been no study on the key issues for many years until early 1970s. In 1970, Lu [13] first emphasized these issues of dS/AdS -space and asked 'why we must use the Minkowski-metric?' A few years later, with his collaborators he began to study the special relativity in dS/AdS -space [14]. Recently, promoted by the observations on the dark universe, the study has been made further [15]-[20].

In fact, there should be three kinds of relativity in the $Mink/dS/AdS$ -space with Poincaré, dS or AdS invariance, respectively, on almost equal footing. The existence of three kinds of special relativity can be understood as the physical counterparts of the three kinds of geometry, the Euclid(-flat), Riemann-spherical and Lobachevsky-hyperbolic geometry, on almost equal footing except the fifth axiom on parallels. In these geometries of constant curvature in 4 dimensions, there are Descartes or Beltrami coordinate systems as a kind of special coordinate systems for Euclidean or non-Euclidean geometry, respectively. In these systems, the points, straight-lines and metric are invariant or mutually transformed under linear transformations of $ISO(4)$ for Euclid-space or the \mathcal{FLT} s of $SO(5)$, $SO(1, 4)$ for Riemann-sphere and Lobachevsky-hyperboloid, respectively. Beltrami [5] introduced such coordinates for description of Lobachevsky-plane first and completed by Klein [6]. Under an inverse Wick rotation [17], these constant curvature spaces become $ISO(1, 3)$, $SO(1, 4)$ and $SO(2, 3)$ -invariant $Mink$, dS and AdS -spacetime, respectively, with events, straight world-lines and Minkowski, or Beltrami-metric of physical signature, say, $(+, -, -, -)$. Thus, Klein's invariance program for geometry under transformation groups [8] should imply the principle of relativity in those maximally symmetric spacetimes.

Einstein's special relativity is set up on the $Mink$ -spacetime as the counterpart of Euclid-space, on the dS/AdS -spacetime as the non-Euclidian counterpart of 4-d Riemann-sphere and Lobachevski-hyperboloid there is just dS/AdS -invariant special relativity, respectively, based on the principle of relativity and the postulate on universal constants of c and R as the curvature radius of the dS/AdS -spacetime.

Thus, there is an inertial law for free particles and light signals in $d\mathcal{S}/\text{AdS}$ -spacetime. It is similar to the inertial law in either Newton mechanics or Einstein's special relativity. Accordingly, as a momentum-‘picture’ in quantum mechanics, there should be an inertial-like law for the ‘group velocity’ of some ‘wave packets’ of free particles in Snyder's model on $d\mathcal{S}/\text{AdS}$ -space of momenta.

It is important and interesting that in terms of the Beltrami model of $d\mathcal{S}$ -space, there is an interchangeable dual relation between Snyder's quantized space-time model [1] as a simplest and earliest DSR [3, 4] and $d\mathcal{S}$ -invariant relativity [13, 14, 15, 16, 17, 18, 19, 20]. In addition, from Snyder's constant a (similar to κ in DSR) or the Planck length ℓ_P and the cosmological constant Λ , it follows a dimensionless constant g

$$\iota := \kappa^2/R^2 \rightarrow g^2 := 3\ell_P^2 R^{-2} \simeq G\hbar c^{-3}\Lambda \simeq 10^{-122}. \quad (1)$$

This dimensionless constant g has been appeared in a kind of simple models of $d\mathcal{S}/\text{AdS}$ -gravity [22, 23, 24]), to characterize the self-interaction of gravity with local $d\mathcal{S}/\text{AdS}$ -invariance, respectively.

Thus, we may propose such a conjectured Planck scale-cosmological constant duality: The cosmological constant should be a fundamental constant together with the Planck constant \hbar , Newton gravitational constant G and c . The physics at such two scales should be dual to each other and linked via gravity of local $d\mathcal{S}$ -invariance characterized by a dimensionless coupling constant g .

Thus, there no longer exist the puzzle on Λ as the ‘vacuum energy’. It should transfer to: What is the origin of the dimensionless constant g ? Is g calculable?

In this talk, we first briefly recall Snyder's model in $d\mathcal{S}$ -space of momenta in section 2. Then we introduce the key issues of the Beltrami model of a Riemann sphere and that of a $d\mathcal{S}$ -space via an inverse Wick rotation in section 3. We briefly introduce the $d\mathcal{S}$ -invariant relativity in section 4. We show the interchangeable relation between Snyder's model and $d\mathcal{S}$ -invariant relativity and propose the Planck scale-cosmological constant duality in section 5. We end with some concluding remarks.

II. SNYDER'S QUANTIZED SPACE-TIME AND DSR

Snyder considered a homogenous quadratic form

$$\begin{aligned} -\eta^2 &= \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 := \eta^{AB}\eta_A\eta_B < 0, \\ (\eta^{AB})_{A,B=0,\dots,4} &= (\eta_{AB}) = \text{diag}(1, -1, -1, -1, -1), \end{aligned} \quad (2)$$

where η_A may be regarded as the homogeneous (projective) coordinates of a real 4-d space of constant curvature, a $d\mathcal{S}$ -space. According to Snyder, this was inspired by Pauli.

Then Snyder defined the energy-momentum with a natural unit of length a

$$p_0 = \frac{1}{a} \frac{\eta_0}{\eta_4}, \quad p_\alpha := \frac{1}{a} \frac{\eta_\alpha}{\eta_4}, \quad \hbar = 1, \quad \alpha = 1, 2, 3. \quad (3)$$

Quantum mechanically, in this ‘momentum picture’ the operators for the time coordinate and the space coordinates denoted as \hat{t}, \hat{x}_α should be given by:

$$\begin{aligned} \hat{x}_\alpha &:= i \left[\frac{\partial}{\partial p_\alpha} + a^2 p_\alpha p_j \frac{\partial}{\partial p_j} \right], \quad j = 0, \dots, 3, \\ \hat{x}_0 &:= i \left[\frac{\partial}{\partial p_0} - a^2 p_0 p_j \frac{\partial}{\partial p_j} \right], \quad x_0 = \hat{ct}. \end{aligned} \quad (4)$$

They are no longer commutative rather noncommutative ‘quantized’ operators. According to Snyder, they form an $so(1.4)$ algebra (in what follows all hats $\hat{}$ are omitted) together with the ‘boost’ M_α and ‘3-angular momentum’ L_α in the space of momenta:

$$\begin{aligned} [x_\alpha, x_\beta] &= ia^2 L_\gamma, \quad [t, x_\alpha] = ia^2 M_\alpha, \\ [L_\alpha, L_\beta] &= \epsilon_{\alpha\beta\gamma} L_\gamma, \quad [M_\alpha, M_\beta] = \epsilon_{\alpha\beta\gamma} M_\gamma; \quad \text{etc.} \end{aligned} \quad (5)$$

Here $L_\alpha = x_\beta p_\gamma - x_\gamma p_\beta$, $M_\alpha = x_\alpha p_0 + x_0 p_\alpha$.

As was mentioned earlier, Snyder's model is a special case of the DSR [3, 4]. In DSR [3], there is a large energy-momentum scale κ near the Planck scale in addition to c for all observers. This scale is just similar to the constant a in Snyder's model.

Some remarks should be made in order. The energy-momentum p_j are inhomogeneous (projective) coordinates and one coordinate patch is not enough to cover the $d\mathcal{S}$ -space of momenta, which can be covered patch-by-patch. Since the 4-d projective space RP^4 is not orientable, in order to preserve the orientation, the antipodal identification should not be taken. Actually, this model can also be realized in terms of the Beltrami model of $d\mathcal{S}$ -space of momenta as was mentioned earlier and will be shown later. In this model, the operators of \hat{x}_j are 4-Killing vectors of the $d\mathcal{S}$ -algebra. Other operators L_α, M_β are just rest 6-Killing vectors forming a homogeneous Lorentz algebra $so(1, 3)$ in the momentum space. Thus, similar to Snyder's model of $d\mathcal{S}$ -space of momenta, a model of \mathcal{AdS} -space of momenta as anti-Snyder's quantized space-time can also be constructed. And the relation between the Snyder-like quantized space-time models and DSR with κ -Poincaré algebra may be described as the Beltrami coordinates for these models and other particular coordinate systems on 4-d $d\mathcal{S}/\mathcal{AdS}$ -space of momenta for other DSR models [4].

It is important that there are also other remarkable issues in the Beltrami model of $d\mathcal{S}/\mathcal{AdS}$ -spaces and we should consider their physical meaning. The most important issue is that in either the Beltrami model or Snyder's projective geometry model, the geodesics are all in linear forms in either Beltrami-coordinates or the inhomogeneous projective coordinates, respectively. Namely, there are *the straight 'world'-lines* in $d\mathcal{S}$ -space of momenta. In fact, in Snyder's approach, these straight 'world'-lines are just the projective straightlines of momenta. Is there any important physical meaning for them? In $d\mathcal{S}$ -space of spacetime, their counterparts are straight world-lines for the test particles and light signals, should their motion be of $d\mathcal{S}$ -invariant inertia? In addition, there is a horizon and according to the standard approach in general relativity there should be some Hawking temperature and entropy [11] in some $d\mathcal{S}$ -spacetimes. What about such kind of properties of the horizon in the $d\mathcal{S}$ -space of momenta? We may keep these issues in mind for the moment.

III. THE BELTRAMI MODEL

A. The Beltrami model of a Riemann sphere

Let us focus on the Beltrami model of a 4-d Riemann sphere \mathcal{S}^4 with positive constant curvature, since its inverse Wick rotation is just the Beltrami model of the $d\mathcal{S}$ -space. Similar issues appear in a 4-d Lobachevski space \mathcal{L}^4 with its inverse Wick rotation as the Beltrami model of the \mathcal{AdS} -space.

The Riemann sphere \mathcal{S}^4 can be embedded in a 5-d Euclid space \mathcal{E}^5

$$\mathcal{S}^4 : \quad \delta_{AB} \xi^A \xi^B = l^2 > 0, \quad A, B = 0, \dots, 4, \quad (6)$$

$$ds_E^2 = \delta_{AB} d\xi^A d\xi^B = d\xi \mathcal{I} d\xi^t, \quad (7)$$

$$\partial_P \mathcal{S}^4 : \quad \delta_{AB} \xi^A \xi^B = 0, \quad (8)$$

where ∂_P is the projective boundary. They are invariant under (linear) rotations of $SO(5)$:

$$\xi \rightarrow \xi' = S \xi, \quad S \mathcal{I} S^t = \mathcal{I}, \quad \forall S \in SO(5). \quad (9)$$

The Beltrami model describes an intrinsic geometry of \mathcal{S}^4 in the Beltrami-space \mathcal{B}_l with Beltrami coordinates:

$$x^i := l \frac{\xi^i}{\xi^4}, \quad \xi^4 \neq 0, \quad i = 0, \dots, 3. \quad (10)$$

To cover $\mathcal{B}_l \sim \mathcal{S}^4$, one Beltrami patch is not enough, but all properties of \mathcal{S}^4 should be well-defined in the Beltrami model patch by patch [16]. For simplicity, we illustrate in one patch only.

The metric (7), the sphere (6) and the boundary (8) restricted on \mathcal{B}_l become the Beltrami-metric, a domain condition and a boundary condition, respectively, as follows:

$$\mathcal{B}_l \quad ds_E^2 = \{\delta_{ij}\sigma_E^{-1}(x) - l^{-2}\sigma_E^{-2}(x)\delta_{il}x^l\delta_{jk}x^k\}dx^i dx^j, \quad (11)$$

$$\sigma_E(x) := \sigma_E(x, x) = 1 + l^{-2}\delta_{ij}x^i x^j > 0, \quad (12)$$

$$\partial_P \mathcal{B}_l : \quad \sigma_E(x) = 0, \quad (13)$$

which are invariant under the \mathcal{FLT} s among Beltrami-coordinates x^i in a transitive form sending the point $A(a^i)$ to the origin $O(o^i = 0)$,

$$\begin{aligned} x^i \rightarrow \tilde{x}^i &= \pm \sigma_E(a)^{1/2} \sigma_E(a, x)^{-1} (x^j - a^j) N_j^i, \\ N_j^i &= O_j^i - l^{-2} \delta_{jk} a^k a^l (\sigma_E(a) + \sigma_E(a)^{1/2})^{-1} O_l^i, \\ O &:= (O_j^i)_{i,j=0,\dots,3} \in SO(4). \end{aligned} \quad (14)$$

There is an invariant for two points $A(a^i)$ and $X(x^i)$ in \mathcal{B}_l , which corresponds the cross ratio among these two points together with the origin and infinity in projective geometry approach:

$$\Delta_{E,l}^2(a, x) = -l^2 [\sigma_E^{-1}(a) \sigma_E^{-1}(x) \sigma_E^2(a, x) - 1]. \quad (15)$$

The proper length between A and B , integral of ds_E over the geodesic segment \overline{AB} :

$$L(a, b) = l \arcsin(|\Delta_E(a, b)|/l). \quad (16)$$

Actually, there is an important property in the Beltrami model: the geodesics of the Beltrami metric are straight-lines in linear form. In fact, the geodesics can be integrated first to get

$$q^i := \sigma_E^{-1}(x) \frac{dx^i}{ds} = \text{consts}. \quad (17)$$

Thus, it is easy to see that the following ratios are constants

$$\frac{q^\alpha}{q^0} = \frac{dx^\alpha}{dx^0} = \text{consts}. \quad (18)$$

One can integrate further to get the linear result:

$$x^i(s) = \alpha^i x^0(s) + \beta^i; \quad \alpha^i, \beta^i = \text{consts}. \quad (19)$$

Under the \mathcal{FLT} s (14) of $SO(5)$, all these properties together with the Beltrami systems are transformed among themselves.

In view of Klein's programm [8], the principle of invariance under symmetry should play a very important role in geometry. There are also other important issues in \mathcal{B}_l analytically. In view of projective geometry, or simply the gnomonic projection, also known as the 'circle-rectilinear' transformation, the Beltrami-coordinates are inhomogeneous projective ones and antipodal identification may not be taken in order to preserve the orientation. The great circles on (6) are mapped to straight-lines, the geodesics (19) in \mathcal{B}_l , and vice versa.

In fact, the Beltrami model of momenta with $l = a$ is just the Euclid version of Snyder's model. What is the physics on $\mathcal{B}d\mathcal{S}_l$ in spacetime? It just leads to the $d\mathcal{S}$ -invariant relativity in $d\mathcal{S}$ -spcetime with $l = R$.

B. The Beltrami model of $d\mathcal{S}$ -space

1. $d\mathcal{S}$ -hyperboloid $H_l \subset M^{1,4}$ and uniform 'great circular' motions

As was noticed [16, 17], via an inverse Wick rotation, the Riemann-sphere \mathcal{S}^4 and its Beltrami model \mathcal{B}_l become the $d\mathcal{S}$ -hyperboloid $H_l \subset M^{1,4}$ and its Beltrami model $\mathcal{B}d\mathcal{S}$, respectively.

Let us first consider the $d\mathcal{S}$ -hyperboloid embedded in a \mathcal{Mink} -space $H_l \subset M^{1,4}$:

$$\begin{aligned} H_l : \quad \eta_{AB} \xi^A \xi^B &= -l^2 < 0, \\ ds^2 &= \eta_{AB} d\xi^A d\xi^B, \quad A, B = 0, \dots, 4, \end{aligned} \quad (20)$$

$$\partial_P H_l : \quad \eta_{AB} \xi^A \xi^B = 0, \quad (21)$$

where $\mathcal{J} = (\eta_{AB}) = \text{diag}(1, -1, -1, -1, -1)$, ∂_P the projective boundary. They are invariant under (linear) transformations of $d\mathcal{S}$ -group $SO(1, 4)$:

$$\xi \rightarrow \xi' = S \xi, \quad S \mathcal{J} S^t = \mathcal{J}, \quad \forall S \in SO(1, 4). \quad (22)$$

Via the inverse Wick rotation, the great circles on S_l should be ‘rotated’ to a kind of uniform ‘great circular’ motions on the $d\mathcal{S}$ -hyperboloid $H_l \subset M^{1,4}$ defined by a conserved 5-d angular momentum:

$$\frac{d\mathcal{L}^{AB}}{ds} = 0, \quad \mathcal{L}^{AB} := m_l \left(\xi^A \frac{d\xi^B}{ds} - \xi^B \frac{d\xi^A}{ds} \right). \quad (23)$$

with an Einstein-like formula for the ‘mass’ m_l

$$-\frac{1}{2l^2} \mathcal{L}^{AB} \mathcal{L}_{AB} = m_l^2, \quad \mathcal{L}_{AB} = \eta_{AC} \eta_{BD} \mathcal{L}^{CD}. \quad (24)$$

There are two ‘time’-like scales on the $d\mathcal{S}$ -hyperboloid, the coordinate-‘time’ ξ^0 and the proper-‘time’s. In order to make sense for the kind of motions, simultaneity should be defined. As in relativity, For a pair of two events $(P(\xi_P), Q(\xi_Q))$, they are simultaneous in the coordinate-‘time’ if and only if

$$\xi_P^0 = \xi_Q^0. \quad (25)$$

A simultaneous 3-hypersurface of $\xi^0 = \text{const}$ is an expanding S^3

$$\begin{aligned} \delta_{ab} \xi^a \xi^b &= R^2 + (\xi^0)^2, \quad a, b = 1, \dots, 4; \\ dl^2 &= \delta_{ab} d\xi^a d\xi^b. \end{aligned} \quad (26)$$

For a kind of ‘observers’ \mathcal{O}_H , it is the same with respect to the proper-‘time’ simultaneity in $H_l \subset M^{1,4}$.

The generators of the $d\mathcal{S}$ -algebra $so(1, 4)$ read:

$$i\hat{\mathcal{L}}_{AB} = \xi_A \frac{\partial}{\partial \xi_B} - \xi_B \frac{\partial}{\partial \xi_A}.$$

They form an $so(1, 4)$ algebraic relation of the $d\mathcal{S}$ -transformations on $H_l \subset M^{1,4}$.

The first Cisimir operator of the algebra is

$$\hat{C}_1 := -\frac{1}{2} l^{-2} \hat{\mathcal{L}}_{AB} \hat{\mathcal{L}}^{AB}, \quad \hat{\mathcal{L}}^{AB} := \eta^{AC} \eta^{BD} \hat{\mathcal{L}}_{CD}, \quad (27)$$

with eigenvalue m_l^2 , which gives rise to the classification of the ‘mass’ m_l .

2. The Beltrami model of $d\mathcal{S}$ -space and uniform motions

The Beltrami model of $d\mathcal{S}$ -space (\mathcal{BdS} -space) is the inverse Wick rotation of the Beltrami model of Riemann sphere.

There exist Beltrami coordinate-systems covering \mathcal{BdS} -space patch by patch. On each patch, there are Beltrami metric, domain condition and boundary condition with $(\eta_{ij})_{ij=0, \dots, 3} = \text{diag}(1, -1, -1, -1)$

$$\mathcal{BdS} : \quad ds^2 = [\eta_{ij} \sigma^{-1}(x) + l^{-2} \eta_{il} \eta_{jk} x^l x^k \sigma^{-2}(x)] dx^i dx^j, \quad (28)$$

$$\sigma(x) := \sigma(x, x) = 1 - l^{-2} \eta_{ij} x^i x^j > 0, \quad (29)$$

$$\partial(\mathcal{BdS}) : \quad \sigma(x) = 0. \quad (30)$$

invariant under \mathcal{FLT} s of $SO(1, 4)$

$$\begin{aligned} x^i \rightarrow \tilde{x}^i &= \pm \sigma^{1/2}(a) \sigma^{-1}(a, x) (x^j - a^j) D_j^i, \\ D_j^i &= L_j^i + l^{-2} \eta_{jl} a^l a^k (\sigma(a) + \sigma^{1/2}(a))^{-1} L_k^i, \\ L &:= (L_j^i)_{i,j=0,\dots,3} \in SO(1, 3). \end{aligned} \quad (31)$$

In such a \mathcal{BdS} , the generators of \mathcal{FLT} s read

$$\begin{aligned} \hat{p}_i &= (\delta_i^j - l^{-2} x_i x^j) \partial_j, \quad x_i := \eta_{ij} x^j, \\ \hat{L}_{ij} &= x_i \hat{p}_j - x_j \hat{p}_i = x_i \partial_j - x_j \partial_i \in so(1, 3), \end{aligned} \quad (32)$$

and form an $so(1, 4)$ algebra

$$\begin{aligned} [\hat{p}_i, \hat{p}_j] &= l^{-2} \hat{L}_{ij}, \quad [\hat{L}_{ij}, \hat{p}_k] = \eta_{jk} \hat{p}_i - \eta_{ik} \hat{p}_j, \\ [\hat{L}_{ij}, \hat{L}_{kl}] &= \eta_{jk} \hat{L}_{il} - \eta_{jl} \hat{L}_{ik} + \eta_{il} \hat{L}_{jk} - \eta_{ik} \hat{L}_{jl}. \end{aligned} \quad (33)$$

Note that for the free particles with ‘mass’ m_l , the uniform ‘great circular’ motions in the $d\mathcal{S}$ -hyperboloid having a set of conserved observables as the 5-d angular momentum with an Einstein-like formula, now should become a kind of uniform ‘motions’ along straight ‘world’-lines in \mathcal{BdS} -space.

C. Snyder’s quantized space-time via Beltrami model

It is clear that Snyder’s model can be reformulated as a \mathcal{BdS} -model of momenta with $l = a$ and \hat{p}_i in (32) as the spacetime coordinates’ operators \hat{x}_j in the ‘momentum picture’ of the quantized space-time and $\hat{L}^\alpha, \hat{M}^\beta$ are just rest 6-generators \hat{L}_{ij} in (33) of Lorentz algebra $so(1, 3)$. Actually, the algebra (5) is the same as (33) in the space of momenta. Similarly, a quantized space-time in \mathcal{AdS} -space of momenta as an anti-Snyder’s model can also be constructed. Thus, the relation between these quantized space-time models and DSR (with κ -Poincaré algebra) may be described as the Beltrami coordinates and other particular coordinate systems on 4-d $d\mathcal{S}/\mathcal{AdS}$ -space of momenta[4].

It is important to note that in these models after an inverse Wick rotation the inverses of ratios in (18) become ‘group velocity’ components of some ‘wave-packets’ and they should be also constants

$$\frac{\partial E}{\partial p^\alpha} = \text{consts.} \quad E = p^0 c, \quad \alpha = 1, 2, 3. \quad (34)$$

Thus, there is a kind of uniform motions with component ‘group velocity’ or a *law of inertia-like* hidden in these Snyder’s models.

Since Snyder’s model is a special case of the DSR, in which there are some noncommutative aspects such as the κ -deformed Poincaré algebra, there should also be some noncommutative properties in the Beltrami model.

D. Klein’s Erlangen program and the principle of relativity

Weakening the Euclid fifth axiom leads to Riemann and Lobachevski geometry on almost equal footing with Euclid. As was emphasized, their physical analogies via an inverse Wick rotation are two kinds of $d\mathcal{S}/\mathcal{AdS}$ -invariant relativity on $d\mathcal{S}/\mathcal{AdS}$ -spacetime, respectively, on the almost equal footing with Einstein’s special relativity on Minkowski spacetime.

In fact, there are one-to-one correspondences between the three kinds of geometry and their physical counterparts via the inverse Wick rotation. We list them as follows:

4-d Geometry	vs.	(1+3)-d Physics
$\mathcal{S}^4/\mathcal{E}^4/\mathcal{L}^4$		$d\mathcal{S}^{1,3}/\mathcal{M}^{1,3}/Ad\mathcal{S}^{1,3}$
$SO(5)/ISO(4)/SO(1,4)$		$SO(1,4)/ISO(1,3)/SO(2,3)$
Points		Events
Straight-line		Straight world-line
Principle of Invariance		Principle of Relativity
Klein		Galilei, Poincaré, de Sitter-Lu
Erlangen Program		Theories of Relativity
...		...

It should be noted that the 4-d Riemann-sphere \mathcal{S}^4 may be regarded as an instanton with an Euler number $e = 2$ in the sense that it is a solution of the Euclidean version of gravitational field equations, its quantum tunnelling scenario should support $\Lambda > 0$ as the \mathcal{BdS} [17]. It will be shown that in the simple model of the $d\mathcal{S}$ -gravity [22, 23, 24] this is the case as in the general relativity.

IV. DE SITTER INVARIANT RELATIVITY

A. Inertial motions, transformations and the principle of relativity

In general, we may define the inertial motions as a kind of the uniform (coordinate) velocity motions along a straight line. Namely, in a (coordinate) system $S(x)$ the motions satisfy

$$x^\alpha = x_0^\alpha + v^\alpha(t - t_0), \quad v^\alpha = \frac{dx^\alpha}{dt} = \text{consts}, \quad \alpha = 1, 2, 3, \quad (35)$$

the motions are called the inertial ones and the system the inertial system (\mathcal{IS}). These are the same as in Newton's mechanics and Einstein's special relativity. The differences among them are that the proper length of a rigid ruler or the proper time of a ideal clock are not assumed to obey the Euclid geometry. In other words, the spacial coordinates themselves and the temporal coordinate itself are not assumed to be uniform. This leads to the transformations among \mathcal{IS} s are different.

In an inertial (coordinate) system S , there is a particle with uniform velocity motion (35). If in the transformed \mathcal{IS} S' , the same particle is described by

$$x'^\alpha = x'_0{}^\alpha + v'^\alpha(t' - t'_0), \quad v'^\alpha = \frac{dx'^\alpha}{dt'} = \text{consts}, \quad (36)$$

what are the most general transformations? Due to Umov, Weyl and Fock [12], the most general form of the transformations should be

$$x'^i = f^i(t, x^\alpha), \quad i = 0, \dots, 3, \quad (37)$$

which transform a uniform straightline motion in S with (35) to a motion of the same nature in S' with (36) are that the four functions f^i are ratios of linear functions, all with the same denominators, i.e. the \mathcal{FLT} s.

Further, we should require that there exist a metric in 4-d spacetime, in which there are inertial systems, and the \mathcal{FLT} s form a group with ten parameters, like the Galilei group in Newton's mechanics and Poincaré group in Einstein's special relativity, including four for spacetime 'translations', three for boosts, and the rest three for space rotations. Thus, according to the properties of maximally symmetric spaces [21]: such kind of 4-d spaces with metric invariant under ten-parameter transformation groups should be maximally symmetric spaces of constant curvature with radius R or $R \rightarrow \infty$. Namely, they are the $d\mathcal{S}/Mink/Ad\mathcal{S}$ space with $SO(1,4)/ISO(1,3)/SO(2,3)$ -invariance, respectively.

Thus, for the $d\mathcal{S}/Ad\mathcal{S}$ -spacetime, the Beltrami systems are just these systems. Therefore, on the $\mathcal{BdS}/\text{anti-}\mathcal{BdS}$ -spacetime, there are also the principle of relativity and the postulate on universal constants. The principle requires that *all physical laws without gravity are invariant under the \mathcal{FLT} s of $d\mathcal{S}/Ad\mathcal{S}$ -group among the inertial systems,*

respectively. The postulate states that in \mathcal{IS} s on 4-d spacetimes, there are two universal constants: the speed of light c and the length R as the curvature radius.

As was mentioned for the Beltrami model of Riemann sphere, one patch of the Beltrami coordinate cannot cover the whole $d\mathcal{S}/Ad\mathcal{S}$ -spacetimes, but the later can be covered by the Beltrami coordinate systems patch by patch and all transition functions on intersections between different patches are of \mathcal{FLT} -type [16].

In each patch, say, $U_4, \xi^4 > 0$, the Beltrami coordinates are

$$x^i|_{U_4} = R \frac{\xi^i}{\xi^4}, \quad i = 0, \dots, 3; \quad \xi^4 = (\xi^{02} - \sum_{a=1}^3 \xi^{a2} + R^2)^{1/2} > 0, \quad (38)$$

there are Beltrami metric (28), condition (29) and boundary (30) with $l = R$ invariant under \mathcal{FLT} s (31) of $SO(1, 4)$ with $l = R$, which transform a point $A(a), \sigma(a) > 0 \in \mathcal{BdS}$ to the origin. It is clear that the $\mathcal{IS} S(x)$ maps to $\mathcal{IS} \tilde{S}(\tilde{x}^i)$ and the inertial motions in S are transformed to that in \tilde{S} . Namely, the geodesics are straight world-lines in linear forms, and vice versa, and transformed among themselves.

Thus, there is the law of inertia in $\mathcal{BdS}/\text{anti-}\mathcal{BdS}$: *The free particles and light signals without undergoing any unbalanced forces should keep their uniform motions along straight world-lines in the linear forms in $\mathcal{BdS}/\text{anti-}\mathcal{BdS}$ -space, respectively.*

The equation of motion for a forced particle can also be given [18].

For free particles there is a set of inertial conserved quantities p^i, L^{ij} along a geodesic,

$$\begin{aligned} p^i &= \sigma(x)^{-1} m_R \frac{dx^i}{ds}, \quad \frac{dp^i}{ds} = 0; \\ L^{ij} &= x^i p^j - x^j p^i, \quad \frac{dL^{ij}}{ds} = 0. \end{aligned} \quad (39)$$

These are the pseudo 4-momentum p^i , pseudo 4-angular-momentum L^{ij} of the particle. They constitute a conserved 5-d angular momentum as was shown in (23) with $l = R$ and satisfy a generalized Einstein formula in \mathcal{BdS} from the Einstein-like formula (24):

$$E^2 = m_R^2 c^4 + p^2 c^2 + \frac{c^2}{R^2} j^2 - \frac{c^4}{R^2} k^2, \quad (40)$$

with energy $E = p^0$, momentum p^α , $p_\alpha = \delta_{\alpha\beta} p^\beta$, ‘boost’ k^α , $k_\alpha = \delta_{\alpha\beta} k^\beta$ and 3-angular momentum j^α , $j_\alpha = \delta_{\alpha\beta} j^\beta$. Note that m_R^2 now is the eigenvalue of 1st Casimir operator of $d\mathcal{S}$ -group, the same as the one in (27) with $l = R$.

If we introduce the Newton-Hooke constant ν [19] and link the curvature radius R with the cosmological constant $R \simeq (3/\Lambda)^{1/2}$

$$\nu := \frac{c}{R} \simeq c(3/\Lambda)^{-1/2}, \quad \nu^2 \sim 10^{-35} s^{-2}. \quad (41)$$

It is very tiny. Thus, local experiments on ordinary scales can not distinguish the $d\mathcal{S}$ -invariant relativity from Einstein’s special relativity.

The interval between two events and thus the light-cone can be well defined as the inverse Wick rotation counterparts of (15) and (16), respectively.

In fact, for two separate events $A(a^i)$ and $X(x^i)$ in \mathcal{BdS} ,

$$\Delta_R(a, x)^2 = R^2 [\sigma(a)^{-1} \sigma(x)^{-1} \sigma(a, x)^2 - 1] \quad (42)$$

is invariant under the \mathcal{FLT} s of $SO(1, 4)$. Thus, the interval between A and B is timelike, null or spacelike, respectively, according to

$$\Delta_R^2(a, b) \gtrless 0. \quad (43)$$

The proper length of time/space-like interval between A and B is the integral of ds over the geodesic segment \overline{AB} :

$$S_{t-like}(a, b) = R \sinh^{-1}(|\Delta(a, b)|/R), \quad (44)$$

$$S_{s-like}(a, b) = R \arcsin(|\Delta(a, b)|/R). \quad (45)$$

The light-cone at A with running points X is

$$\mathcal{F}_R := R\{\sigma(a, x) \mp [\sigma(a)\sigma(x)]^{1/2}\} = 0. \quad (46)$$

It satisfies the null-hypersurface condition. At the origin $a^i = 0$, the light cone becomes a Minkowski one and c is numerically the velocity of light in the vacuum.

There is a horizon tangent to the boundary in \mathcal{BdS} for the observers \mathcal{O}_I :

$$\lim_{a \rightarrow a'} \sigma(a, x) = 0, \quad \lim_{a \rightarrow a'} \sigma(a) = 0. \quad (47)$$

B. Two kinds of simultaneity, the principle of relativity and cosmological principle

In order to make physical measurements, one should define simultaneity. Different from Einstein's special relativity, there are two kinds of simultaneity related to two kinds of measurements, or to the principle of relativity and the cosmological principle, respectively, in $d\mathcal{S}/Ad\mathcal{S}$ -spacetime. In the contraction $R \rightarrow \infty$, they coincide with each other.

1. The Beltrami-time simultaneity

Let us first consider the Beltrami coordinate simultaneity, called the Beltrami simultaneity. For inertial observers \mathcal{O}_I at spacial origin, two events (A, B) are simultaneous if and only if

$$a^0 := x^0(A) = x^0(B) =: b^0. \quad (48)$$

It defines a $1 + 3$ decomposition of \mathcal{BdS} -space

$$ds^2 = N^2(dx^0)^2 - h_{ab}(dx^a + N^a dx^0)(dx^b + N^b dx^0) \quad (49)$$

with lapse function, shift vector and induced 3-geometry on 3-hypersurface Σ_c in one coordinate patch, respectively

$$\begin{aligned} N &= \{\sigma_{\Sigma_c}(x)[1 - (x^0/R)^2]\}^{-1/2}, \\ N^a &= x^0 x^a [R^2 - (x^0)^2]^{-1}, \\ h_{ab} &= \delta_{ab} \sigma_{\Sigma_c}^{-1}(x) - [R \sigma_{\Sigma_c}(x)]^{-2} \delta_{ac} \delta_{bd} x^c x^d, \\ \sigma_{\Sigma_c}(x) &= 1 - (x^0/R)^2 + \delta_{ab} x^a x^b / R^2. \end{aligned} \quad (50)$$

It is easy to see that at $x^0 = 0$, $\sigma_{\Sigma_c}(x) = 1 + \delta_{ab} x^a x^b / R^2$, $N = \sigma_{\Sigma_c}^{-1/2}(x)$, $N^a = 0$. Then the Cauchy hypersurface is $\Sigma_c \simeq S^3$. And at Beltrami time $x^0 \neq 0$, as long as x^0 is still time-like, we should also have $\Sigma_c \simeq S^3$.

The Beltrami simultaneity leads to the definition of Beltrami ruler and its relation to the spacial distance of two events. A Beltrami non-Euclidean ruler at time x^0 is given by

$$dl_B^2|_{x^0} = -h_{ab}|_{x^0} dx^a dx^b. \quad (51)$$

2. The Proper-time simultaneity and the Robertson-Walker-like coordinates

The proper time τ of a rest clock on the time axis of Beltrami system, $\{x^a = 0\}$, reads

$$\tau = R \sinh^{-1}(R^{-1} \sigma^{-\frac{1}{2}}(x) x^0). \quad (52)$$

one can naturally define the second simultaneity by means of τ as the proper-time simultaneity. The events are simultaneous with respect to the proper time of a clock rest at the origin of the Beltrami spatial coordinates if and only if these events corresponding to the same τ

$$x^0 \sigma^{-1/2}(x, x) = (\xi^0 :=) R \sinh(\tau/R) = \text{const.} \quad (53)$$

If τ is taken as a ‘cosmic’ temporal coordinate together with the spatial Beltrami coordinates, the Beltrami system becomes a Robertson-Walker-like system with a metric:

$$\begin{aligned} ds^2 &= d\tau^2 - dl_C^2 = d\tau^2 - \cosh^2(\tau/R) dl_0^2, \\ dl_0^2 &= \{\delta_{ab} \sigma_{\Sigma_\tau}^{-1}(x) - [R \sigma_{\Sigma_\tau}(x)]^{-2} \delta_{ac} \delta_{bd} x^c x^d\} dx^a dx^b, \\ \sigma_{\Sigma_\tau}(x, x) &= 1 + R^{-2} \delta_{ab} x^a x^b > 0, \end{aligned} \quad (54)$$

with dl_0^2 is a 3-dimensional Beltrami-metric on an S^3 with radius R . It is an ‘empty’ accelerated expanding cosmological model with a slightly closed cosmos in $O(R)$.

There is an important prediction different from the ‘standard model’ if the dark universe is asymptotically a dS , the 3-d cosmic space is asymptotically an expanding S^3 in Robertson-Walker- dS spacetime. If we take $R^2 \simeq 3\Lambda^{-1}$, the deviation from the flatness is in $O(\Lambda)$.

Due to the relation between the principle of relativity and cosmological principle, dS -space provides also such a model that the dS -cosmic background with cosmological constant just acts as *the origin of the inertial law*. This supports as a base the principle of relativity in Beltrami-coordinates. Of course, the precondition is that the maximum symmetry ensures the existence of the inertial motions and these two principles in dS -spacetime. In other words, the dS -group as a maximum symmetry assures that there are the principle of relativity and cosmological principle together with their relation in the dS -spacetime as a maximally symmetric one. Thus, the Robertson-Walker- dS -cosmos with cosmological constant and other cosmic objects including the distant stars as test stuffs should just display as the origin of dS -inertial law in the Beltrami-coordinates on dS -spacetime.

In fact, in dS -space there are a type of *inertial-comoving observers* having a kind of two-time-scale timers with respect to the Beltrami-time and the cosmic-time. This reflects that there is an important relation linking the principle of relativity with the cosmological principle of dS -invariance.

Inertial \mathcal{O}_I	vs.	Co-moving \mathcal{O}_C
Beltrami-systems		Robertson-Walker- dS -systems
Beltrami timer		Cosmic-time timer
Beltrami ruler		Co-moving ruler
Inertial observables		Co-moving observables
...		...

Thus, what should be done for those inertial-comoving observers is merely to switch their timers from cosmic-time back to Beltrami-time and vice versa. Namely, once the observers would carry on the experiments in their laboratories, they should take their timers switching on Beltrami-time and off the cosmic-time so as to act as inertial observers and all observations are of inertial. When they would take cosmic-observations on the distant stars and the cosmic objects other than the cosmological constant as test stuffs they may just switch off the Beltrami-time and on the cosmic-time again, then they should act as a kind of comoving observers as they hope.

C. On temperature and entropy

In general relativity, the Hawking-temperature and entropy at the dS -horizon [11] lead to the dS -entropy puzzle.

There is another explanation now [20]. Eq. (52) shows the imaginary Beltrami-time has no periodicity, since both Beltrami-time axis and its imaginary counterpart are straight-lines without coordinate acceleration, but the imaginary proper-time has such a period that is inversely proportional to the Hawking-temperature $(2\pi R)^{-1}$ at the horizon. If the temperature Green’s function can still work, this should indicate that the horizon in Beltrami-coordinates is at zero-temperature and needless to introduce entropy.

Since there is no gravity, the Hawking-temperature and (area-)entropy S at the horizon in other coordinate-systems should be non-inertial effects [20]. Such a kind of non-inertial thermodynamic properties for dS -spacetime are similar to those in Rindler metric in flat spacetime.

Similar issues may also be introduced and make sense in the dS -space of momenta.

V. THE PLANCK SCALE - Λ DUALITY

A. An interchangeable relation and the duality

It is interesting to see that there is an interchangeable dual relation between Snyder's model and $d\mathcal{S}$ -invariant relativity in \mathcal{BdS} with l :

Snyder's QST	$d\mathcal{S}$ -invariant relativity
momentum 'picture'	coordinate 'picture'
\mathcal{BdS} -space of momenta	\mathcal{BdS} -spacetime
$l = a \sim$ Planck length	$l = R \sim$ cosmic radius
constant 'group velocity'	constant 3-velocity
quantized space-time	'quantized' momenta
\hat{x}_α, \hat{t}	\hat{p}_α, \hat{E}
$\tilde{T}_p = 0$, without \tilde{S}_p	$T = 0$, without S

It is important that from two fundamental constants, the Planck length $l_P := (G\hbar c^{-3})^{1/2}$ and the $d\mathcal{S}$ -radius $R \simeq (3/\Lambda)^{1/2}$, it follows a dimensionless constant

$$\iota := \kappa/R \rightarrow g := \sqrt{3}\ell_P/R, \quad g^2 \simeq G\hbar c^{-3}\Lambda \sim 10^{-122}. \quad (55)$$

Since there is Newton constant, g should describe the gravity and its self-interaction with local $d\mathcal{S}$ -invariance between these two scales.

Thus, these indicate that there should be a Planck scale-cosmological constant duality: *The cosmological constant is a fundamental scale as the Planck length. The physics at such two scales should be dual to each other in some 'phase' space and linked via the gravity with local $d\mathcal{S}$ -invariance characterized by the dimensionless constant g .*

It is clear that the Planck scale- Λ duality could also be proposed for the \mathcal{AdS} -space. And this is a kind of 'ultraviolet-infrared' relations since the cosmological constant and the Planck length provide an IR and a UV cut-off, respectively. The 'UV-IR duality' (or connection, etc) appeared in various cases already. For example, in the sense of \mathcal{AdS}/CFT correspondence and holographic principle [25, 26]. The relations among these UV-IP should be investigated.

B. Is the dimensionless constant calculable?

It is interesting to see that g^2 is in the same order of difference between Λ and the theoretical quantum 'vacuum energy', there is no longer this puzzle in view of the $d\mathcal{S}$ -invariant relativity and gravity with local $d\mathcal{S}$ -invariance. However, since Λ a fundamental constant as c, G and \hbar , a further question should be: What is the origin of the dimensionless constant g ? Is it calculable? This is just the first question of the 'top ten' [27]: 'Are all the (measurable) dimensionless parameters that characterize the physical universe calculable in principle or are some merely determined by historical or quantum mechanical accident and uncalculable?'

It is important to note that there are some hints on the answer for this dimensionless constant g . First, among 4-d Euclid, Riemann and Lobachevski spaces there is only the Riemann-sphere with non-vanishing 4-d topological number. Thus, the quantum tunneling scenario for the Riemann-sphere \mathcal{S}^4 as an instanton of gravity to the $d\mathcal{S}$ -space may explain why the cosmological constant should be positive, i.e. $\Lambda > 0$. We should show in the next subsection that in a simple model of $d\mathcal{S}$ -gravity in $d\mathcal{S}$ -Lorentz gauge shows that this is just the case. Further, if the action of the $d\mathcal{S}$ -gravity is of the Yang-Mills type, then its Euclidean version is of a non-Abelian type with local $so(5)$ symmetry. Thus, due to asymptotic freedom, the gauge-coupling constant, say g , should be running and approaching to zero as the momentum reaches to infinity. However, for the case of gravity, the momentum could not be reaching to infinity but the Planck scale as a fixed point so that the Euclidean counterpart of the dimensionless coupling constant should be very tiny and link Λ with the Planck scale.

C. A simple model of $d\mathcal{S}$ -gravity

In general relativity, there is no room for special relativity in $d\mathcal{S}/Ad\mathcal{S}$ -space. In $d\mathcal{S}/Ad\mathcal{S}$ -invariant relativity, there is no gravity in $d\mathcal{S}/Ad\mathcal{S}$ -space. How to describe gravity?

As the spirit of Einstein's equivalence principle, the gravity should be based on localized special relativity. However, in general relativity there are only local Lorentz-frames of homogeneous Lorentz-symmetry without localized translations.

In Einstein equation $\mathbf{G} = 8\pi G \mathbf{T}$ symbolically (see, e.g. [28]), the Einstein-Cartan 'moment of rotation' \mathbf{G} is related to local homogeneous Lorentz rotation, while the stress-energy tensor \mathbf{T} is concerning the translations in \mathcal{Mink} -space (see, e.g. [29]). Why geometry is connected with matter in different symmetry?

As an enhanced equivalence principle with localization of special relativity, there should be *the localization principle: On spacetimes with gravity, there always exist local relativity-frames with local Poincaré/ $d\mathcal{S}/Ad\mathcal{S}$ -symmetry, physical laws must take the gauge covariant versions of their special-relativistic forms with respect to the local Poincaré/ $d\mathcal{S}/Ad\mathcal{S}$ -symmetry, respectively.* If geometry and matter are connected in same local symmetry, there should be some gauge-like dynamics for gravity.

A simple model for the $d\mathcal{S}$ -gravity has shown such a feature. Its action is gauge-like and characterized by a dimensionless constant g . In the $d\mathcal{S}$ -Lorentz gauge, it reads [22, 23]

$$\begin{aligned} S_G &= -\frac{1}{4g^2} \int_{\mathcal{M}} d^4x \varepsilon (\mathcal{F}_{jk}^{AB} \mathcal{F}_{AB}^{jk}) \\ &= \int_{\mathcal{M}} d^4x \varepsilon \left(\frac{1}{16\pi G} (F - 2\Lambda) - \frac{1}{4g^2} F_{jk}^{ab} F_{ab}^{jk} + \frac{1}{32\pi G} T_{jk}^a T_a^{jk} \right), \end{aligned} \quad (56)$$

where $\varepsilon = \det(e_j^a)$, \mathcal{F}_{jk}^{AB} is the $d\mathcal{S}$ -curvature of a $d\mathcal{S}$ -connection $\mathcal{B}_j^{AB} \in so(1, 4)$, with $\mathcal{B}_j^{ab} = B_j^{ab}$, $\mathcal{B}_j^{a4} = R^{-1}e_j^a$, F , F_{jk}^{ab} and T_{jk}^a are scalar curvature, curvature and torsion of the Riemann-Cartan manifolds \mathcal{M} with Lorentz frame e_j^a and connection B_j^{ab} . Namely,

$$\mathcal{B} := \mathcal{B}_j dx^j, \quad \mathcal{B}_j := (\mathcal{B}_j^{AB})_{A,B=0,\dots,4} = \begin{pmatrix} B_j^{ab} & R^{-1}e_j^a \\ -R^{-1}e_j^b & 0 \end{pmatrix} \in so(1, 4), \quad (57)$$

where R is the $d\mathcal{S}$ -curvature radius. The curvature valued in the $d\mathcal{S}$ -algebra reads:

$$\mathcal{F}_{jk} = (\mathcal{F}_{jk}^{AB}) = \begin{pmatrix} F_{jk}^{ab} + R^{-2}e_j^{ab} & R^{-1}T_{jk}^a \\ -R^{-1}T_{jk}^b & 0 \end{pmatrix} \in so(1, 4), \quad (58)$$

where $e_{bjk}^a = e_j^a e_{bk} - e_k^a e_{bj}$, $e_{bj} = \eta_{ab} e_j^a$, F_{jk}^{ab} and T_{jk}^a are given by

$$T_{jk}^a = \partial_j e_k^a - \partial_k e_j^a + B_{cj}^a e_k^c - B_{ck}^a e_j^c, \quad (59)$$

$$F_{bjk}^a = \partial_j B_{bk}^a - \partial_k B_{bj}^a + B_{cj}^a B_{bk}^c - B_{ck}^a B_{bj}^c. \quad (60)$$

where $B_{bj}^a = \eta_{ac} B_j^{ac}$. And $F = \frac{1}{2} F_{jk}^{ab} e_{ab}^{jk}$.

It is important that this model can pass all tests for general relativity and may provide a more suitable platform for the precise cosmology.

VI. CONCLUDING REMARKS

With plenty of $d\mathcal{S}$ -puzzles, the dark universe as an accelerated expanding, asymptotic to $d\mathcal{S}$ -space with a tiny cosmological constant Λ [30, 31] greatly challenges Einstein's theory of relativity as foundation of the physics in large scale.

Symmetry, its localization and symmetry breaking play extreme important roles in physics. For the space-time and gravity physics, the maximum symmetry and its localization should also play an extreme important role. There

should be three kinds of relativity and their contractions [19], and three kinds of theory of gravity as localization of corresponding relativity with a gauge-like dynamics and their contractions, respectively. Our Nature should chose one of them.

Via an interchangeable dual relation between Snyder's-like quantized space-time models in dS/AdS -space of momenta and the dS/AdS -invariant special relativity in dS/AdS -spacetime with the Beltrami coordinates, respectively, there should also be a duality in the physics at the Planck scale and cosmological constant. And between these two scales is the gravity based on the localization of corresponding relativity with a gauge-like dynamics of full localized symmetry characterized by a dimensionless coupling constant.

The dark universe may already indicate that the dS -invariant relativity and the gravity with local dS -invariance should be the foundation of physics in large-scale.

Needless to say, there is still long way to go!

Acknowledgments

The author would like to thank Professor/Dr. Q.K. Lu, Z. Chang, C.-G. Huang, Y. Tian, Z. Xu and B. Zhou for valuable discussions. This work is partly supported by NSFC under Grant Nos 10375087, 90503002.

-
- [1] H.S. Snyder, Quantized space-time, *Phys. Rev.* **71** (1947) 38.
 - [2] K. Greisen, *Phys. Rev. Lett.* **16** (1966), 748; G. T. Zatsepin and V. A. Kuz'min, *JETP Lett.* **4** (1966), 78.
 - [3] G. Amelino-Camelia, *Int. J. Mod. Phys. D* **11**, 35 (2002); *Phys. Lett. B* **510**, 255 (2001).
 - [4] J. Kowalski-Glikman, *Phys. Lett. B* **547** (2002) 291; J. Kowalski-Glikman and S. Nowak, *Class. Quant. Grav.* **20** (2003) 4799.
 - [5] E. Beltrami, *Opere Mat.*, **1** (1868) 374-405.
 - [6] F. Klein, *Gör. Nachr.* (1871) 419.
 - [7] B. A. Rosenfeld, A history of non-euclidean geometry: evolution of the concept of a geometric space (New York, 1987).
 - [8] F. Klein (1872), The Erlangen programm, *Math. Intelligence*, **22** (1977).
 - [9] W. de Sitter, Einstein's theory of gravitation III, *Roy. Astr. Soc. Month. Not.* **78** (1917) 3.
 - [10] W. Pauli, *Theory of Relativity*, Pergamon 1958.
 - [11] G. W. Gibbons and S. W. Hawking, *Phys. Rev. D* **15**, 2738 (1977).
 - [12] V. Fock, *The Theory of Space-Time and Gravitation*, Pergamon 1964. And references therein.
 - [13] K.H. Look (Q.-K. Lu) 1970, Why the Minkowski metric must be used? unpublished.
 - [14] K.H. Look, C.L. Tsou (Z.L. Zou) and H.Y. Kuo (H.-Y. Guo), *Acta Phy. Sin.* **23** (1974) 225; *Nature* (Shanghai, Suppl.) (in Chinese); H.-Y. Guo, *Kexue Tongbao (Chinese Sci. Bull.)* **22** (1977) 487 (in Chinese); in *Proc. 2nd Marcel Grossmann Meet. on GR*, ed. by R. Ruffini, (North-Holland) 1982, 801; *Nucl. Phys. B* (Proc. Suppl.) **6** (1989) 381.
 - [15] Q. K. Lu, Dirac's conformal spaces and de Sitter spaces, in memory of the 100th anniversary of Einstein special relativity and the 70th anniversary of Dirac's de Sitter spaces and their boundaries. MCM-Workshop series. **1** 2005.
 - [16] H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, *Mod. Phys. Lett. A* **19** 1701 (2004); *Phys. Lett. A* **331** (2004) 1; hep-th/0403171.
 - [17] H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, *Chinese Phys. Lett.* **22** (2005) 2477; hep-th/0508094.
 - [18] H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, *Acta Phy. Sin.* **54** (2005) 2494.
 - [19] C.-G. Huang, H.-Y. Guo, Y. Tian, Z. Xu and B. Zhou, hep-th/0405137; Y. Tian, H.-Y. Guo, C.-G. Huang, Z. Xu and B. Zhou, *Phys. Rev. D* **71** (2005) 044030; hep-th/0411004.
 - [20] H.-Y. Guo, C.-G. Huang and B. Zhou, *Europhys. Lett.* **72** (2005) 1045; hep-th/0404010.
 - [21] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, (1963). Wiley, New York: Interscience.
 - [22] Y.S. Wu, G.D. Li and H.-Y. Guo, *Kexue Tongbao (Chinese Sci. Bull.)* **19**(1974)509; H.-Y. Guo, *ibid* **21** (1976) 31; I. An, S. Chen, Z.L. Zou and H.-Y. Guo, *ibid* 379.
 - [23] P.K. Townsend, *Phys. Rev. D* **15** (1977) 2795. S.W. MacDowell and F. Mansouri, *Phys. Rev. Lett.* **38** (1977) 739; 1376. K.S. Stelle and P.C. West, *Phys. Rev. D* **21** (1980) 1466. A.A. Tseytlin, *Phys. Rev. D* **26**, (1982) 3327. F. Wilczek, *Phys. Rev. Lett.* **80** (1998) 4951.

- [24] M.L. Yan, B.H. Zhao and H.-Y. Guo, *Kexue Tongbao (Chinese Sci. Bull.)* 24 (1979) 587; *Acta Physica Sinica* (In Chinese) 33 (1984) 1377; 1386.
- [25] L. Susskind and E. Witten, The Holographic Bound in Anti-de Sitter Space, hep-th/9805114.
- [26] A. W. Peet and J. Polchinski, UV-IR relations in AdS dynamics, *Phys. Rev.* **D59**, 065011 (1999).
- [27] D. Gross, The future of physics, January 2005. <http://agenda.cern.ch/fullAgenda.php?>
- [28] C. Misner, K. Thorne and J.A. Wheeler, *Gravitation*, W.H. Free. & Com., 1972.
- [29] A. Trautman, *GRG* **3** (1972) 167.
- [30] C. L. Bennett, et al, *Astrophys. J.* (Suppl.) 148 (2003) 1, ArXiv: astro-ph/0302207; D. N. Spergel *et al.* ArXiv: astro-ph/0603449.
- [31] A.G. Riess et al, *Astro. J.* **116**, 1009 (1998); S. Perlmutter et al, *Astrophys. J.* **517**, 565 (1999).